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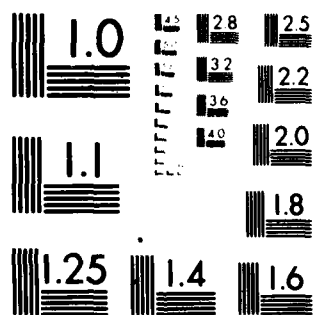
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PREPOSTERIOR ANALYSIS OF BAYES ESTIMATORS OF MEAN LIFE, I

by  
RICHARD E. BARLOW  
and  
ALEXANDER S. WU

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PREPOSTERIOR ANALYSIS OF BAYES ESTIMATORS OF MEAN LIFE, I

by

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FEBRUARY 1980

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## SUMMARY

17 In considering life test experiments from a behavioristic Bayes point of view, we may speculate on the effect of model misspecification on the posterior mean under alternative stopping rules. Of course, under the life distribution model, we expect the mean of the posterior to equal the mean of the prior. However, if we think the model may have been misspecified, we would no longer expect this equality. We show that for Bayes estimators of mean life under the exponential model and general priors on mean life, we expect, based on a preposterior analysis, that the mean of the posterior will increase with sample size when the true model has an increasing failure rate and certain fixed stopping rules are used. Also, we show that for Bayes estimators of mean life based on the natural conjugate prior for the exponential model and complete samples, the Bayes risk is less when the coefficient of variation is less than for the exponential model. Recommendations relative to use of life test sampling plans based on an exponential model are made.

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# PREPOSTERIOR ANALYSIS OF BAYES ESTIMATORS OF MEAN LIFE, I

by

Richard E. Barlow and Alex Wu

## 1. INTRODUCTION

In this paper we consider selected life test sampling plans with respect to life distribution model robustness. Our objective is to obtain results useful in selecting life test sampling plans. To this end, let  $F(\cdot | \theta)$  be a life distribution with mean life  $\theta \in \Theta = [0, \infty)$ . Since  $\theta$  is unknown to us, we model our uncertainty about  $\theta$  by letting  $\tilde{\theta}$  denote a random variable with density  $\pi(\theta)$ . We suppose  $\pi$  is chosen and fixed. It is important to remark that  $\theta$  and  $\pi$  may be considered independently of our life distribution model  $F(\cdot | \theta)$  subject only to measurability restrictions. In some parts of this paper we will concentrate on the prior density

$$\pi(\theta) = \frac{b^a \theta^{-(a+1)} e^{-b/\theta}}{\Gamma(a)} \quad (1.1)$$

where  $a, b > 0$ . Later, we also assume  $a > 2$ . This is the *inverted gamma* prior which also happens to be the natural conjugate prior when

$$F(x | \theta) = 1 - \exp(-x/\theta).$$

With this setup, suppose we choose and perform a life test experiment,  $E$ , and observe data,  $D$ . After specifying a life distribution,  $F(\cdot | \theta)$ , and prior  $\pi$ , we may compute,  $\pi(\theta | D)$ , the posterior density for  $\theta$  given the data,  $D$ . Let expected mean square error

be our criterion for merit of an estimator of  $\theta$ . Then we seek  $\hat{\theta}(D)$  such that

$$E_{\pi, F}(\tilde{\theta} - \hat{\theta}(D))^2$$

is minimum. Expectation is with respect to the probability measure on the sample space, determined by our sampling plan, and  $\theta$ , induced by our life distribution model  $F(\cdot | \theta)$  and prior density  $\pi$ . It is well known that

$$\hat{\theta}(D) = E(\tilde{\theta} | D) = \int_{\Theta} \theta \pi(\theta | D) d\theta, \quad (1.2)$$

the mean of the posterior density,  $\pi(\theta | D)$ , minimizes expected mean square error among estimators based on the data.  $E(\tilde{\theta} | D)$  is called the *Bayes estimator* of  $\theta$  with respect to squared error loss. Also, it is well known that

$$\hat{\theta}(D) = E(\tilde{\theta} | D)$$

is Bayesian unbiased; i.e.,

$$E_{\pi, F}(\hat{\theta}(D)) = E_{\pi, F} \int_{\Theta} \theta \pi(\theta | D) d\theta = \theta_0 \quad (1.3)$$

where  $\theta_0$  is the mean of the prior density,  $\pi$ . That is, we expect, preposterior analysis, the posterior mean to be the same as the mean of the prior. Since we have not actually observed any data, this is perfectly consistent. Hence, given  $\pi$ ,  $F(\cdot | \theta)$  and our experiment  $E$ , the mean of the posterior is Bayesian unbiased and has least expected mean square error among estimators of  $\theta$  based on the data,  $D$ .

Suppose, however, that we are committed to using a certain life distribution model; namely the exponential

$$G(x \mid \theta) = 1 - \exp(-x/\theta) .$$

This often happens in industry. A contract may specify the life distribution model and procedures to be used to demonstrate a reliability goal. If this is the case, we may *speculate*, prior to observing data, about the effect of alternative life distribution models on our estimator. For experiment  $E$ , let

$$\hat{\theta}_G(D) = E(\tilde{\theta} \mid D)$$

be our Bayes estimator for  $\theta$  where the posterior is computed using the exponential life distribution model,  $G(\cdot \mid \theta)$ . If  $G$  is incorrect and  $F(\cdot \mid \theta)$  is a more accurate model, then in general,

$$E_{\pi, F}\{\hat{\theta}_G(D)\} \neq \theta_0 ,$$

i.e., our estimator, based on  $G$ , is not necessarily unbiased if observations come from  $F(\cdot \mid \theta)$ . Also,

$$E_{\pi, F}\{\tilde{\theta} - \hat{\theta}_F(D)\}^2 \leq E_{\pi, F}\{\tilde{\theta} - \hat{\theta}_G(D)\}^2 \quad (1.4)$$

so that neither is  $\hat{\theta}_G(D)$  best with respect to expected mean square error. Our problem is to evaluate the extent of these discrepancies with respect to the sampling plan chosen, i.e., the number tested and the stopping rule employed, as well as for alternative classes of life distributions. We call this paper a preposterior analysis because we

are in effect speculating about the posterior distribution *before* any data are actually observed. We hope to be able to recommend which sampling plan to employ when the model, say the exponential, is actually misspecified.

At the data analysis stage with the data in hand, we might pursue a Bayesian robustness study relative to our life distribution model. For example, we might consider a class of life distributions  $F(\cdot \mid \alpha, \theta)$  indexed by mean life,  $\theta$ , and a parameter  $\alpha$ . For  $\alpha = 1$ , we might have

$$F(x \mid \alpha = 1, \theta) = 1 - e^{-(x/\theta)}.$$

Then, for a joint prior  $\pi(\alpha, \theta)$ , we compute the marginal posterior density

$$\pi^*(\theta \mid D) = \int \pi(\alpha, \theta \mid D) d\alpha$$

and

$$\pi(\theta \mid D) = \pi(\theta \mid \alpha = 1, D).$$

If the posterior densities, or their means, are "close" we then conclude that our procedures based on  $\alpha = 1$ , and our data  $D$ , are robust for the family  $F(\cdot \mid \alpha, \theta)$ , Box and Tiao (1972, Chapter 3). We do *not* pursue this type of robustness here. In contrast, our speculations and analysis are all *in the mind*. Our specified prior,  $\pi$ , is our only reference with respect to knowledge about the mean life,  $\theta$ .

Our main results concern Bayesian bias of  $\hat{\theta}(D)$  under selected stopping rules when we conjecture that  $F(\cdot | \theta)$  may actually have decreasing mean residual life conditional on  $\theta$ . See Theorems 3.1 and 4.1. Our second criterion is the expected mean square error of our estimator when  $F(\cdot | \theta)$  is *not* exponential. Results for this criterion are given in Sections 2 and 3. This criterion is explored in greater depth in Part II.

## 2. COMPLETE SAMPLES

In this section, we suppose  $n$  units are life tested until failure so that our data  $D = (x_1, x_2, \dots, x_n)$  consists of  $n$  observed lifetimes. Given  $F(\cdot | \theta)$ , these are assumed independent conditional on  $\theta$ . If the random lifetime,  $X$ , has distribution

$$G(x | \theta) = 1 - \exp(-x/\theta),$$

then  $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$  is clearly sufficient for  $\theta$ . Let  $\pi$  be the natural conjugate prior given by (1.1) with  $a > 2$ . Then, the Bayes estimator is

$$\hat{\theta}(D) = (1 - w)\theta_0 + w\bar{x} \quad (2.1)$$

where

$$\theta_0 = b/(a - 1), \quad w = \gamma^2 / \left\{ \frac{1}{n} E \text{Var}(X | \bar{\theta}) + \gamma^2 \right\}$$

and

$$\gamma^2 = b^2/(a - 1)^2(a - 2).$$

We assume henceforth, that  $a > 2$  so that  $\gamma^2 < \infty$ . For the prior given by (1.1),  $w = n/(a + n - 1)$ .

It can be shown, (Bühlmann, 1970) that (2.1) is actually the least squares linear, in observations, approximation to the mean of the posterior density for any  $F(\cdot | \theta)$  and any prior  $\pi$  with mean  $\theta_0$  and variance  $\gamma^2$ . For the exponential distribution  $G(\cdot | \theta)$ , (2.1) is

the mean of the posterior *only* if  $\pi$  is the natural conjugate prior (Diaconis and Ylvisaker, 1979).

Clearly, for *any*  $F(\cdot | \theta)$  and  $\pi$ ,

$$E_{\pi, F}\{\hat{\theta}(D)\} = (1 - w)\theta_0 + w\theta_0 = \theta_0 \quad (2.2)$$

so that (2.1) is always Bayes unbiased. Also, for any  $F(\cdot | \theta)$  and prior  $\pi$ ,

$$E_{\pi, F}\{\tilde{\theta} - \hat{\theta}(D)\}^2 = (1 - w)^2 \gamma^2 + \frac{w^2}{n} E \text{Var}(X | \tilde{\theta}) \quad (2.3)$$

where  $X$  has distribution  $F(\cdot | \theta)$ . Since

$$\lim_{n \rightarrow \infty} w = \lim_{n \rightarrow \infty} \frac{\gamma^2}{\frac{1}{n} E \text{Var}(X | \tilde{\theta}) + \gamma^2} = 1,$$

it follows that the limiting expected mean square error is zero for all  $F(\cdot | \theta)$  and  $\pi$ . Hence, sample size  $n$  can be chosen to achieve any specified expected mean square error given  $F(\cdot | \theta)$  and  $\pi$ .

Let  $CV(X | \theta) = \sqrt{\text{Var}(X | \theta)}/\theta$ , the *coefficient of variation* of  $X$ . Let  $Y$  have distribution  $G(x | \theta) = 1 - \exp(-x/\theta)$ .

THEOREM 2.1. If  $CV(X | \theta) \leq (>) 1$  for all  $\theta \in \Theta$ , then

$$E_{\pi, F}\{\tilde{\theta} - \hat{\theta}(D)\}^2 \leq (>) E_{\pi, G}\{\tilde{\theta} - \hat{\theta}(D)\}^2, \quad (2.4)$$

where  $\hat{\theta}(D)$  is given by (2.1).

Proof. For the exponential distribution  $G(\cdot | \theta)$ ,  $CV(Y | \theta) = 1$  for all  $\theta$ . Hence, if  $CV(X | \theta) \leq (>) 1$ , then

$$\text{Var}(X \mid \tilde{\theta}) \leq (\geq) \tilde{\theta}^2$$

implies

$$E \text{Var}(X \mid \tilde{\theta}) \leq (\geq) E \tilde{\theta}^2 = E \text{Var}(Y \mid \tilde{\theta}) . \quad (2.5)$$

(2.4) now follows from (2.5) and (2.3). ||

From Theorem 2.1, it is clear that if  $F(\cdot \mid \theta)$  is a member of any of the classes IFR, IFRA, NBU or NBUE, see Barlow and Proschan (1975), then the expected mean square error for  $\hat{\theta}(D)$  is less than that for the exponential model. Hence, the sample size  $n$  can be determined assuming model exponentiality to meet an expected mean square error criterion and it follows that the expected mean square error will be even less if  $F(\cdot \mid \theta)$  is in any of the previous classes. Of course, if  $F$  is DFR, DFRA, NWU or NWUE, the expected mean square error will be greater than in the exponential case.

The *Bayes risk* is the expected mean square error with respect to  $F(\cdot \mid \theta)$  and  $\pi$  when  $\hat{\theta}(D)$  is the posterior mean based on using  $G(\cdot \mid \theta)$ . From (2.4), we see easily that  $CV(X \mid \theta) \leq 1$  implies the Bayes risk under  $F(\cdot \mid \theta)$  and  $\pi$  is less than the Bayes risk under  $G$  since

$$\begin{aligned} \min_a E_{\pi, F}(\tilde{\theta} - a)^2 &= E_{\pi, F}\{\tilde{\theta} - E(\tilde{\theta} \mid D)\}^2 \\ &\leq E_{\pi, G}\{\tilde{\theta} - \hat{\theta}(D)\}^2 \end{aligned}$$

where  $E(\tilde{\theta} \mid D)$  is computed with respect to the posterior distribution based on  $F(\cdot \mid \theta)$ .

### 3. OBSERVATION UNTIL THE $k^{\text{TH}}$ FAILURE

In this section, we suppose that  $n$  units are life tested until the  $k^{\text{th}}$  observed failure occurs,  $k$  is fixed in advance. We call this sampling plan (a). At this point, testing stops. Although this sampling plan is perhaps not as useful as the plan which stops at a preassigned time  $t$ , it is easier to study. Also, results for this stopping rule suggest similar results for the case of time truncated sampling, sampling plan (b).

Our data consists of the  $k$  ordered lifetimes  $D = \{x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(k)}\}$ . If the random lifetime  $X$  has distribution

$$G(x | \theta) = 1 - \exp(-x/\theta),$$

then the total time on test,  $T = \sum_{i=1}^k x_{(i)} + (n - k)x_{(k)}$ , is sufficient for  $\theta$ . Let  $\pi$  be the natural conjugate (1.1). Then, the Bayes estimator is

$$\hat{\theta}(k) = (1 - w)\theta_0 + w \frac{T}{k} \quad (3.1)$$

where  $w = \gamma^2 / \left\{ \frac{1}{k} E \text{Var}(X | \tilde{\theta}) + \gamma^2 \right\}$ . Since, under exponentiality and conditional on  $\theta$ ,  $T$  is the sum of  $k$  independent exponentials, (3.1) is actually the least squares linear, in  $T$ , approximation to the mean of the posterior density for any prior  $\pi$  with mean  $\theta_0$  and variance  $\gamma^2$  as in Section 2.

Although (3.1) is Bayes unbiased for the exponential model and *any* prior, it is *not* necessarily unbiased for arbitrary  $F(\cdot | \theta)$ .

THEOREM 3.1. If  $F(\cdot | \theta)$  has decreasing mean residual life conditional on  $\theta$ , then

$$E_{\pi, F}\{\hat{\theta}(k)\} \geq \theta_0. \quad (3.2)$$

The prior is the natural conjugate (1.1). Compare this with Theorem 4.1 in the next section. The inequality is reversed if  $F(\cdot | \theta)$  has increasing mean residual life.

Proof. Assume  $F(\cdot | \theta)$  has decreasing mean residual life. For  $k = n$ ,

$$\hat{\theta}(n) = \frac{(a-1)}{a+n-1} \theta_0 + \frac{n}{a+n-1} \bar{x}$$

and  $E_{\pi, F}\theta(n) = \theta_0$ . Applying Bayes Theorem and properties of the natural conjugate, we have for  $k < n$ ,

$$\hat{\theta}(n) = (1 - \hat{w})\hat{\theta}(k) + \hat{w} \frac{T\{x_{(k)}, \infty\}}{n - k} \quad (3.3)$$

where  $\hat{w} = \frac{n-k}{a+n-1}$  and  $T\{x_{(k)}, \infty\}$  is the total time on test from  $x_{(k)}$  to the last ordered failure time. Taking expectations, we have

$$\theta_0 = \left(\frac{a+k-1}{a+n-1}\right) E\hat{\theta}(k) + \left(\frac{n-k}{a+n-1}\right) E\theta^*\{x_{(k)}\} \quad (3.4)$$

where  $\theta^*\{x_{(k)}\}$  is the mean residual life conditional on  $\theta$  and given survival to  $x_{(k)}$ . By assumption,  $\theta^*\{x_{(k)}\} \leq \theta$  for all  $x_{(k)}$  and  $\theta$  so that

$$E_{\pi, F}\theta^*\{x_{(k)}\} \leq \theta_0.$$

Hence, from (3.4), we must have

$$E_{\pi, F} \hat{\theta}(k) \geq \theta_0 .$$

The proof when  $F(\cdot | \theta)$  has decreasing mean residual life is similar. ||

We call the class of failure distributions with decreasing mean residual life DMRL. If  $F(\cdot | \theta)$  is IFR, then  $F(\cdot | \theta)$  is DMRL. This is *not* true in general if IFR is replaced by IFRA. However, a useful comparison is available under star ordering ( $\leq_*$ ) of distribution functions. See Barlow and Proschan (1975, Chapter 4.)

Remark: If  $F \leq_* G$  and  $G$  is exponential, then  $F$  is IFRA.

THEOREM 3.2. If  $F_1(\cdot | \theta) \leq_* F_2(\cdot | \theta) \forall \theta \in \Theta$ , then

$$E_{\pi, F_1} \{\hat{\theta}(k)\} \geq E_{\pi, F_2} \{\hat{\theta}(k)\} . \quad (3.5)$$

$\hat{\theta}(k)$  is given by (3.1) and is calculated under  $G$ . This strengthens (3.2) relative to star ordering.

Proof. From (3.1),  $\hat{\theta}(k) = (1 - w)\theta_0 + w \frac{T}{k}$  and

$$w = \gamma^2 / \left\{ \frac{1}{n} E \text{Var}(x | \tilde{\theta}) + \gamma^2 \right\}$$

is fixed and

$$T = \sum_{i=1}^k x_{(i)} + (n - k)x_{(k)} .$$

It follows from Barlow and Proschan (1966) that

$$E_{F_1}(T \mid \theta) \geq E_{F_2}(T \mid \theta) .$$

Hence,  $E_{\pi, F_1}(T) \geq E_{\pi, F_2}(T)$  . Applying this result to (3.1), the theorem follows. ||

Under IFR assumptions, Bayesian bias actually increases with  $n$  for fixed  $k$  . We prove this and more in

THEOREM 3.3. If  $F(\cdot \mid \theta)$  is IFR, then

$$E_{\pi, F}\{\hat{\theta}(k) \mid n\}$$

increases in  $n$  for fixed  $k$  and *arbitrary* prior.  $\hat{\theta}(k) = E(\tilde{\theta} \mid k, T)$  and the prior is arbitrary with mean  $\theta_0$  and variance  $\gamma^2$  . Note that  $\hat{\theta}(k)$  is the posterior mean under  $G$  and arbitrary prior. It is *not* necessarily linear in  $T$  .

Proof. Assume  $F$  is IFR. It can be shown that when  $\pi$  is arbitrary, the posterior mean under  $G$

$$E(\tilde{\theta} \mid k, T)$$

is increasing in  $T$  for fixed  $k$  . In Barlow and Proschan (1966), it is shown that when  $F(\cdot \mid \theta)$  is IFR,

$$P(T > t \mid k, n, \theta)$$

is increasing in  $n$  for fixed  $k$  and conditional on  $\theta$  . It follows that

$$E_{\pi, F}(\hat{\theta}(k) | n) = E_{\pi, F}(E(\tilde{\theta} | k, T, n))$$

is increasing in  $n$  . ||

If  $F(\cdot | \theta)$  is DFR, then  $E_{\pi, F}(\hat{\theta}(k) | n)$  decreases in  $n$  for fixed  $k$  . For the natural conjugate prior and  $\hat{\theta}(k)$  given by (3.1),  $E_{\pi, F}(\hat{\theta}(n)) = \theta_0$  . It follows that if  $n_1 < n_2$  and  $F(\cdot | \theta)$  is IFR, then

$$E_{\pi, F}(\hat{\theta}(k) | n_2) \geq E_{\pi, F}(\hat{\theta}(k) | n_1) \geq E_{\pi, F}(\hat{\theta}(k) | k = n) = \theta_0 .$$

All this is true for fixed  $k$  and the natural conjugate prior (1.1).

If  $F(\cdot | \theta)$  is DFR, all inequalities are reversed.

### 3.1 The Expected Mean Square Error Criterion

For sampling plan (a), not only the Bayesian bias but also the expected mean square error can be unfavorable for  $\hat{\theta}(k)$  , given by (3.1). In particular, if  $F(x | \theta)$  has order of contact at the origin, with respect to  $x$  , greater than that of an exponential distribution  $G(x | \theta)$  , then

$$\lim_{n \rightarrow \infty} E_{\pi, F}(\tilde{\theta} - \hat{\theta}(k))^2 = \infty$$

for fixed  $k$  . For example, this is true if

$$F(x | \theta) = \begin{cases} 0 & x < x_0 \\ 1 - \exp\left\{\frac{-(x - x_0)}{(\theta - x_0)}\right\} & x \geq x_0, \end{cases} \quad (3.6)$$

where  $\theta \geq x_0 > 0$ .

THEOREM 3.4. Assume either  $F(x_0 | \theta) = 0$  for  $x_0 > 0$  or

$$\lim_{x \rightarrow 0} \frac{F(x | \theta)}{x^\alpha} = c > 0 \quad (3.7)$$

where  $\alpha > 1$ . That is,  $F(x | \theta)$  is  $O(x^\alpha)$  as  $x \rightarrow 0$ . Then, for  $\hat{\theta}(k)$ , given by (3.1) and fixed  $k$ ,

$$\lim_{n \rightarrow \infty} E_{\pi, F}\{\tilde{\theta} - \hat{\theta}(k)\}^2 = \infty. \quad (3.8)$$

Proof. It is easy to verify that

$$E_{\pi, F}\{\tilde{\theta} - \hat{\theta}(k)\}^2 = \text{Var}_{\pi, F}\{\tilde{\theta} - \hat{\theta}(k)\} + E_{\pi, F}^2\{\tilde{\theta} - \hat{\theta}(k)\}. \quad (3.9)$$

Hence, it is sufficient to show for fixed  $k$  that

$$\lim_{n \rightarrow \infty} E_{\pi, F}^2\{\tilde{\theta} - \hat{\theta}(k)\} = \infty.$$

Since  $T = nx_{(1)} + (n-1)(x_{(2)} - x_{(1)}) + \dots + (n-k+1)(x_{(k)} - x_{(k-1)})$ ,

it is enough to show

$$\lim_{n \rightarrow \infty} nE(X_{1n} \mid \theta) = \infty$$

for all  $\theta$ .

Suppose  $F(x_0 \mid \theta) = 0$  and  $x_0 > 0$ . Then, obviously

$$\lim_{n \rightarrow \infty} nE(X_{1n} \mid \theta) \geq \lim_{n \rightarrow \infty} nx_0 = \infty.$$

From condition (3.7), it follows that  $F(\cdot \mid \theta)$  is in the domain of attraction of a Weibull distribution with shape parameter  $\alpha > 1$ .

See Barlow and Proschan (1975, p. 241). It follows that

$$\lim_{n \rightarrow \infty} P\{(cn)^{1/\alpha} X_{1n} > x \mid \theta\} = \exp\{-(\lambda x)^\alpha\} \text{ for all } x \geq 0$$

where  $\theta = \Gamma(1 + 1/\alpha)/\lambda$ . Hence,

$$\begin{aligned} \lim_{n \rightarrow \infty} E\{(cn)^{1/\alpha} X_{1n} \mid \theta\} &= \lim_{n \rightarrow \infty} \int_0^\infty P\{(cn)^{1/\alpha} X_{1n} > x \mid \theta\} dx \\ &= \int_0^\infty e^{-(\lambda x)^\alpha} dx = \Gamma(1 + 1/\alpha)/\lambda = \theta \end{aligned}$$

since the convergence is uniform in  $x$ . Therefore,

$$\lim_{n \rightarrow \infty} nE(X_{1n} \mid \theta) = \lim_{n \rightarrow \infty} n^{1-1/\alpha} \theta/c^{1/\alpha} = \infty$$

since  $\alpha > 1$  by assumption. ||

For general  $F$ , we have

$$\begin{aligned}
E_{\pi, F}(\tilde{\theta} - \hat{\theta}(k))^2 &= w^2 \left[ \text{Var}_{\pi} \left\{ E\left(\frac{T}{k} \mid \tilde{\theta}\right) \right\} + E_{\pi} \left\{ \text{Var}\left(\frac{T}{k} \mid \tilde{\theta}\right) \right\} \right] \\
&- 2w \text{Cov}_{\pi} \left\{ E\left(\frac{T}{k} \mid \tilde{\theta}\right), \tilde{\theta} \right\} + \text{Var}_{\pi}(\tilde{\theta}) + w^2 \left[ E_{\pi} E\left(\frac{T}{k} \mid \tilde{\theta}\right) - \theta_0 \right]^2.
\end{aligned} \tag{3.10}$$

Now let  $F(x \mid \theta) = 1 - \exp\{-(\lambda x)^{\alpha}\}$  for  $\alpha > 0$  and  $\theta = \frac{\Gamma(1+1/\alpha)}{\lambda}$ .

From (3.10) and for  $k = 1$ , we have when  $0 < \alpha < 1$ ,

$$\lim_{n \rightarrow \infty} E_{\pi, F}(\tilde{\theta} - \hat{\theta}(1))^2 = \text{Var}_{\pi}(\tilde{\theta}) + w^2 \theta_0^2.$$

It follows that for  $k = 1$  and the Weibull distribution with  $\alpha > 0$ ,

$$\lim_{n \rightarrow \infty} E_{\pi, F}(\tilde{\theta} - \hat{\theta}(1))^2 \geq \lim_{n \rightarrow \infty} E_{\pi, G}(\tilde{\theta} - \hat{\theta}(1))^2.$$

Hence, asymptotically, for  $k = 1$ , the expected mean square error for the Weibull distribution is always greater than or equal, asymptotically, to the expected mean square error in the exponential case.

Suppose  $k = n = 1$ , then for  $\alpha > 1$ ,

$$E_{\pi, F}(\tilde{\theta} - \hat{\theta}(1))^2 \leq E_{\pi, G}(\tilde{\theta} - \hat{\theta}(1))^2.$$

Hence, for  $\alpha > 1$  and  $k = 1$ , the expected mean square error as a function of  $n$  is initially less than the corresponding expected mean square error for the exponential case. It can be shown that eventually for some  $n$  it is greater and remains greater thereafter. In the limit, of course, it is infinite, Theorem 3.4.

4. OBSERVATION UNTIL TIME  $t$ 

In this section, we suppose  $n$  units are life tested until time  $t$ . This is a common and important life test sampling plan, called sampling plan (b). If the random lifetime  $X$  has distribution

$$G(x | \theta) = 1 - \exp(-x/\theta)$$

then  $k$ , the observed number of failures and  $T = \sum_{i=1}^k x_{(i)} + (n-k)t$ ,

the total time on test are together sufficient statistics for  $\theta$ .

For the natural conjugate prior (1.1), with  $a > 1$ , the mean of the posterior is now

$$\hat{\theta}(t) = \begin{cases} (1-w)\theta_0 + w \frac{T}{k} & \text{for } k \geq 1 \\ \theta_0 + \frac{T}{a-1} & \text{for } k = 0 \end{cases} \quad (4.1)$$

where  $w = k/(a+k-1)$ . Under exponentiality and the natural conjugate prior,

$$E_{\pi, G}\{\hat{\theta}(t)\} = \theta_0 \quad (4.2)$$

by (1.3) since the posterior mean is always Bayesian unbiased when expectation is computed with respect to the model. However, in general,  $E_{\pi, F}\{\hat{\theta}(t)\} \neq \theta_0$ .

**THEOREM 4.1.** If  $F(\cdot | \theta)$  has decreasing mean residual life, i.e., is DMRL, then

$$E_{\pi, F}\{\hat{\theta}(t)\} \geq \theta_0 \quad \forall t > 0. \quad (4.3)$$

$\pi$  is the natural conjugate (1.1) for the exponential. The inequality is reversed if  $F(\cdot | \theta)$  has increasing mean residual life.

Proof. Assume  $F(\cdot | \theta)$  has decreasing mean residual life. If  $t = \infty$ , then we have complete observations and our estimator becomes

$$\hat{\theta} = (1 - w)\theta_0 + w\bar{x}$$

and  $E_{\pi, F}\hat{\theta} = \theta_0$  for all  $F(\cdot | \theta)$  in this case, see Section 2. Now suppose  $k$  failures occur in  $[0, t)$ . By Bayes' Theorem and properties of the natural conjugate prior,

$$\hat{\theta} = \begin{cases} (1 - \hat{w})\hat{\theta}(t) + \frac{\hat{w}T(t, \infty)}{n - k} & k < n \\ \hat{\theta}(t) & k = n \end{cases} \quad (4.4)$$

where  $\hat{w} = \frac{n - k}{a + n - 1}$  and  $T(t, \infty)$  is the total residual life of the remaining  $n - k$  units at time  $t$ .

Let  $D_1 = \{k, T(t)\}$  be the observed data for the interval  $[0, t)$ ; i.e., the number of failures,  $k$ , and the total time on test  $T(t)$  in  $[0, t)$ . Since  $F(\cdot | \theta)$  has decreasing mean residual life conditional on  $\theta$ , we have

$$E_{\pi, F}\left\{\hat{w} \frac{T(t, \infty)}{n - k} \mid D_1\right\} = E_{\pi, F}\left\{\frac{T(t, \infty)}{a + n - 1} \mid D_1\right\} \leq \frac{(n - k)}{(a + n - 1)} E(\tilde{\theta} \mid D_1). \quad (4.5)$$

Hence,

$$E_{\pi, F} \left\{ \hat{w} \frac{T(t, \infty)}{n - k} \mid D_1 \right\} \leq \hat{w} E(\tilde{\theta} \mid D_1) = \hat{w} \hat{\theta}(t) .$$

Note that, given  $D_1$ , the distribution of  $\theta$  is *always* updated using  $G$ . Hence,  $E(\tilde{\theta} \mid D_1) = \hat{\theta}(t)$ . Also,  $\hat{w} = 0$  if  $k = n$ .

It follows that

$$\theta_0 = E_{\pi, F} \hat{\theta} \leq E_{\pi, F} \{(1 - \hat{w}) \hat{\theta}(t)\} + E_{\pi, F} \{\hat{w} \hat{\theta}(t)\}$$

and

$$\theta_0 \leq E_{\pi, F} \hat{\theta}(t)$$

as was to be shown.

All inequalities are reversed when  $F(\cdot \mid \theta)$  has increasing mean residual life. ||

Remarks. If  $F(\cdot \mid \theta)$  is IFR, then it is also DMRL so that (4.3) holds in this case. Since  $F(\cdot \mid \theta)$  IFRA does *not* in general imply  $F(\cdot \mid \theta)$  is DMRL, we do not know at this time whether or not  $F(\cdot \mid \theta)$  IFRA implies (4.3).

The classical sample theory approach to the exponential model and sampling plan (b) is very unsatisfactory (Bartholomew 1957, 1963, and Barlow and Proschan 1967). The principle difficulty is that if no failures occur in  $[0, t)$ , the MLE is infinite. The sample distribution properties of the MLE are also highly unsatisfactory. In the Bayesian approach, the Bayes estimator (4.1) is Bayesian unbiased under exponentiality. There is no sample theory analogue to Theorem 4.1.

## 5. CONCLUDING REMARKS

For complete sampling, there is no Bayesian bias using (2.1). Also, the expected mean square error is finite. For distributions in the classes IFRA and/or NBUE, DFRA and/or NWUE, the preposterior expected mean square error is less than under exponentiality. Hence, for complete sampling and  $F(\cdot | \theta)$  IFRA and/or NBUE, we may proceed under the exponential model assumption. We can, by appropriate choice of sample size, in this case control the preposterior expected mean square error.

However, for the sampling plan which stops at the  $k^{\text{th}}$  observation, or time  $t$ , we have positive Bayesian bias if  $F(\cdot | \theta)$  is DMRL. Hence, if it is necessary to use stopping rule (a) or (b), and we have strong reason to believe that  $F(\cdot | \theta)$  is *not* exponential, then we should consider a Bayesian robustness study at the data analysis stage, see Introduction. The more IFRA or DFRA in the sense of star ordering, see Barlow and Proschan (1975, Chapter 4), the more compelling the reason for a robustness study at the data analysis stage if we are committed to using an exponential model based estimator.

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